The Riemann mapping theorem.

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Theorem (Riemann) Let $\Omega \subsetneq \mathbb{C}$ be a simply connected region; $z_0 \in \Omega$. Then there exists unique conformal bijection $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$, $f'(z_0) > 0$ (i.e. $f'(z_0) \in [0, \infty)$).

Stated by Riemann in 1851. Used the solution of Dirichlet problem, which was incomplete. Moreover, it could only work for domains with piecewise smooth boundaries. Osgood worked with general case in 1900. Köbe proved in 1912.

$$\min_{g|_{\partial\Omega} = f} \iint_\Omega |\nabla g|^2 \, dx\, dy$$

Proof of uniqueness

$f_1, f_2 : \Omega \to \mathbb{D}$, $f_1(z_0) = f_2(z_0) = 0$, $f_1'(z_0) > 0$, $f_2'(z_0) > 0$.

Consider $\varphi := f_1 \circ f_2^{-1} : \mathbb{D} \to \mathbb{D}$ - bijection, conformal. So Möbius.

$\varphi(0) = f_1 \circ f_2^{-1}(0) = f_1(z_0) = 0$, $\varphi'(0) = f_1'(f_2^{-1}(z_0)) \cdot (f_2^{-1})'(0) = \dfrac{f_1'(z_0)}{f_2'(z_0)} > 0 \Rightarrow \varphi(z) = z \Rightarrow$

$f_1(f_2^{-1}(z)) = z$, or $f_1 \equiv f_2$. ∎

Existence, step 1.

Lemma $\exists\, h : \Omega \to \mathbb{D}$ - conformal injection, $h(z_0) = 0$, $h'(z_0) > 0$.

(i.e. $h(\Omega) \subset \mathbb{D}$).

Proof.

First square root trick: Let $w \notin \Omega$. $\phi(z) := z - w \neq 0 \ \forall z \in \Omega$.

Ω - simply connected. So $\exists\, h_1 : \Omega \to \mathbb{C}$: $h_1^2(z) = z - w \ \forall z \in \Omega$. $h_1(z) \neq 0$.

h_1 is conformal ($h_1(z_1) = h_1(z_2) \Rightarrow z_1 - w = h_1^2(z_1) = h_1^2(z_2) = z_2 - w \Rightarrow z_1 = z_2$).

$\forall z \in \Omega$: $-h_1(z) \notin h_1(\Omega)$ (because $-h_1(z) \in h_1(\Omega) \Rightarrow \exists\, z_2 \in \Omega$: $h_1(z_2) = -h_1(z)$, $z_2 \neq z$ [$h_1(z) \neq 0$]

but $z_2 - w = h_1^2(z_2) = (-h_1(z))^2 = z - w \Rightarrow z_2 = z$ - contradiction).

Now: $h_1(z_0) \in h_1(\Omega) \Rightarrow \exists\, r > 0$ $B(h_1(z_0), r) \subset h_1(\Omega)$ (open map).
$h_1(\Omega)$ - region.

So $B(-h_1(z_0), r) \cap h_1(\Omega) = \emptyset \Longleftrightarrow \forall z \in \Omega \ |h_1(z) + h_1(z_0)| > r$.

Consider $h_2(z) = \dfrac{r}{h_1(z) + h_1(z_0)}$. Then h_2 is conformal: $h_2 = \psi \circ h_1$,

where $\psi(w) = \dfrac{r}{w + h(z_0)}$

Also, $|h_2(z)| = \dfrac{r}{|h_1(z) + h_1(z_0)|} < 1 \ \forall z \in \Omega$.

Reminder: for $c \in \mathbb{D}$, $S_c := \dfrac{z - c}{\bar{c} z - 1}$ - Möbius bijection of $\mathbb{D} \to \mathbb{D}$,

$S_c(0) = c$, $S_c(c) = 0$; so $S_c \circ S_c(0) = 0$, $S_c \circ S_c(c) = c$ and $S_c \circ S_c(\infty) = \infty$ - preserves inversion.

so $S_c \circ S_c = \mathrm{id}$.

Consider $h_3(z) := S_{h_2(z_0)} \circ h_2(z)$.

Then h_3 is conformal, $h_3(z_0) = S_{h_2(z_0)}(h_2(z_0)) = 0$.

$\forall z \in \Omega$ $|h_3(z)| < 1$ ($S_{h_2(z_0)}$ maps $\mathbb{D} \to \mathbb{D}$, $\forall z: h_2(z) \in \mathbb{D}$).

Finally: h_3 is conformal, so $h_3'(z_0) \neq 0$.

consider $h(z) := \frac{|h_3'(z_0)|}{h_3'(z_0)} \cdot h_3(z)$. Then $h'(z_0) = |h_3'(z_0)| > 0$

$h(z_0) = h_3(z_0) = 0$.

$h: \Omega \to \mathbb{D}$. ∎

Let $\mathcal{F} := \{h: \Omega \to \mathbb{D}$ – conformal, $h(z_0) = 0$, $h'(z_0) > 0\}$.

$\mathcal{F} \neq \emptyset$ (by step 1)

Maximization idea: find $f \in \mathcal{F}$ such that it maximises certain quantity in $\mathcal{F}$.

Version 1 (Ostrovsky, 1930) Find $f \in \mathcal{F}$ such that $f'(z_0) = \sup_{h \in \mathcal{F}} h'(z_0)$

Version 2 (Koebe, refined by Caratheodory in 1929). Fix $z_1 \neq z_0$ and find $f \in \mathcal{F}$: $|f(z_1)| = \sup_{h \in \mathcal{F}} |h(z_1)|$.

Ahlfors does Version 1.

We'll do Version 2.

Step 2. $\exists f \in \mathcal{F}: |f(z_1)| = \sup_{h \in \mathcal{F}} |h(z_1)|$.

Proof. Let $M := \sup_{h \in \mathcal{F}} |h(z_1)| \leq 1$.

Take $(h_n) \subset \mathcal{F}$: $|h_n(z_1)| \to M$.

$\mathcal{F}$ is uniformly bounded (by 1), so $\exists$ locally uniformly convergent on Ω subsequence (h_{n_k}), $f := \lim_{k \to \infty} h_{n_k}$.

Then $f(z_0) = \lim_{k \to \infty} h_{n_k}(z_0) = 0$

$|f(z_1)| = M \neq |f(z_0)|$ so f is not constant.

By Hurwitz Theorem, f is conformal.

So $f'(z_0) \neq 0$, and $f'(z_0) = \lim_{k \to \infty} h_{n_k}'(z_0) > 0$.

so $f \in \mathcal{F}$, $|f(z_1)| = M$ ∎

Second quadratic trick:

Define $j(z) := z^2$, $\varphi_c(z) := S_{c^2} \circ j \circ S_c(z)$.

(φ_c is not conformal, $\varphi_c'(c) = 0$) but $\varphi_c: \mathbb{D} \to \mathbb{D}$ (each map does it)

$\varphi_c(0) = S_{c^2} \circ j \circ S_c(0) = S_{c^2}(j(c)) = S_{c^2}(c^2) = 0$. $\varphi_c'(0) > 0$.

So $\forall z \in \mathbb{D}$: $|\varphi_c(z)| < |z|$ – by Schwarz Lemma.

Step 3. Let $h \in \mathcal{F}$, $c^2 \notin h(\Omega)$. Then $\exists \tilde{h} \in \mathcal{F}$:

Step 3. Let $h \in \mathcal{F}$, $c^2 \notin h(\Omega)$. Then $\exists\, \tilde{h} \in \mathcal{F}$:

$$h(z) = \varphi_c \circ \tilde{h}(z). \qquad |h(z_1)| < |\tilde{h}(z_1)|$$

Proof. Observe that $S_{c^2} \circ h(z) \neq 0 \ \forall z \in \Omega$ (since $h(z) \neq c^2$).

Then, since Ω is simply connected, $\exists\, g: \Omega \to \mathbb{D}$:

$g^2 = S_{c^2} \circ h(z)$ (i.e. $g(z) = \sqrt{S_{c^2}(h(z))}$). Pick a branch with $g(z_0) = \sqrt{S_{c^2} \circ h(z_0)} = \sqrt{c^2} = c$.

Define: $\tilde{h}(z) := S_c \circ g(z)$. $|\tilde{h}(z)| < 1 \ \forall z \in \Omega$.

Then $\varphi_c \circ \tilde{h}(z) = S_{c^2} \circ g \circ \underbrace{S_c \circ S_c}_{= Id} \circ g(z) = S_{c^2}((g(z))^2) = S_{c^2}(S_{c^2} \circ h(z)) = h(z)$

Also $\tilde{h}(z_0) = S_c \circ g(z_0) = S_c(c) = 0$ $\quad \tilde{h}'(z_0) > 0$.

So $\tilde{h} \in \mathcal{F}$ ∎

Step 4. f constructed in Step 2 is holomorphic bijection $f: \Omega \to \mathbb{D}$.

Proof. Assume that $f(\Omega) \neq \mathbb{D}$. Then $\exists\, w_0 \notin f(\Omega)$, $|w_0| < 1$.

$\exists\, c \in \mathbb{D}$: $c^2 = w_0$. $\quad c^2 \notin f(\Omega)$.

Then, by step 3, $\exists\, \tilde{f} \in \mathcal{F}$: $f(z) = \varphi_c(\tilde{f}(z))$.

Reminder: $\forall w \in \mathbb{D}$; $|\varphi_c(w)| < |w|$

In particular, for $w = \tilde{f}(z_1)$: $\quad |f(z_1)| < |\tilde{f}(z_1)|$ — contradiction!

$\left(|f(z_1)| = \sup_{h \in \mathcal{F}} |h(z_1)| \right)$ ∎

Hyperbolic distance for simply-connected regions.

Def. Let $\Omega \subsetneq \mathbb{C}$ be a simply-connected region.

Let $\varphi: \Omega \to \mathbb{D}$ be a conformal bijection.

Pseudo-hyperbolic distance between $w_1, w_2 \in \Omega$ is defined as:

$$\rho_\Omega(w_1, w_2) \stackrel{def}{:=} \rho_{\mathbb{D}}(\varphi(w_1), \varphi(w_2)) = \frac{|\varphi(w_1) - \varphi(w_2)|}{|1 - \overline{\varphi(w_1)}\varphi(w_2)|}$$

Hyperbolic distance is defined as

$$\ell_{H,\Omega}(w_1, w_2) = \operatorname{arctan} \rho_\Omega(w_1, w_2) = \ell_H(\varphi(w_1), \varphi(w_2)) = \inf_{\substack{\gamma \subset \Omega \\ \gamma \text{ from } w_1 \text{ to } w_2}} \int_\gamma \frac{|\varphi'(\xi)|}{1 - |\varphi(\xi)|^2} |d\xi|$$

Theorem. ρ_Ω and $\ell_{H,\Omega}$ do not depend on the choice of conformal bijection.

Proof. Let $\varphi_1, \varphi_2: \Omega \to \mathbb{D}$ be conformal bijections.

Then $\varphi_1 \circ \varphi_2^{-1} : \mathbb{D} \to \mathbb{D}$ is a conformal bijection, so it is a Möbius map.

$\rho_{\mathbb{D}}$ and l_{H} are conserved by Möbius maps preserving Π w

We also just proved:

<u>Theorem</u>. If $\varphi_1, \varphi_2 : \Omega \to \mathbb{D}$ – conformal bijections then $\exists \theta, a$: $\varphi_1(z) = \varphi_2(S_{\theta,a}(z))$, where $S_{\theta,a}(z) = e^{i\theta} \frac{z+a}{1+\bar{a}z}$.

<u>Theorem</u> (General Schwarz Lemma).

Let $f: \Omega_1 \to \Omega_2$, $f \in A(\Omega_1)$.

Then $\forall z, w \in \Omega_1$, $\rho_{\Omega_2}(f(z), f(w)) \le \rho_{\Omega_1}(z, w)$

Equality $\Leftrightarrow$ f is conformal bijection between Ω_1 and Ω_2.

<u>Proof</u>.

$$\begin{tikzcd} \Omega_1 \arrow[r, "f"] \arrow[d, "\varphi_1"] & \Omega_2 \arrow[d, "\varphi_2"] \\ \mathbb{D} \arrow[r, "\tilde{f}"'] & \mathbb{D} \end{tikzcd}$$

Let $\tilde{f} := \varphi_2 \circ f \circ \varphi_1^{-1} : \mathbb{D} \to \mathbb{D}$.

Then $\rho_{\Omega_2}(f(z), f(w)) = \rho_{\mathbb{D}}(\varphi_2(f(z)), \varphi_2(f(w)))$

$\rho_{\Omega_1}(z, w) = \rho_{\mathbb{D}}(\varphi_1(z), \varphi_1(w))$

So, by Schwarz Lemma, $\rho_{\mathbb{D}}(\tilde{f}(\varphi_1(z)), \tilde{f}(\varphi_2(z))) \le \rho_{\mathbb{D}}(\varphi_1(z), \varphi_1(w))$.

Equality $\Leftrightarrow$ $\tilde{f}$ is conformal bijection $\Leftrightarrow$ f is conformal bijection $\blacksquare$

<u>Solving Dirichlet Problem.</u>

Let Ω be a simply connected region, $\partial\Omega = \gamma$ – simple curve.

Constantin Carathéodory

Theorem (Caratheodory). Let $\varphi : \Omega \to \mathbb{D}$ be a conformal bijection. Then $\exists\ \hat{\varphi} : \overline{\Omega} = \Omega \cup \gamma \longrightarrow \overline{\mathbb{D}}$ – continuous bijection, such that $\forall z \in \Omega\ \ \hat{\varphi}(z) = \varphi(z)$.

Theorem. Let $f \in C(\gamma)$. Then $\exists\ u \in Harm(\Omega) \cap C(\overline{\Omega})$: $\forall z \in \gamma\ \ u(z) = f(z)$ (Dirichlet problem is solvable).

Proof. Let $\hat{\varphi} : \overline{\Omega} \to \overline{\mathbb{D}}$ be continuous bijection with $\hat{\varphi}|_{\Omega}$ – conformal.

Define $\tilde{f} := f \circ \hat{\varphi}^{-1}$ – a continuous function on $\mathbb{T}$.

By Schwarz theorem,

$$\tilde{u}(w) := P_{\tilde{f}}(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|w-e^{it}|^2}\,\tilde{f}(e^{it})\,dt \in Harm(\mathbb{D}) \cap C(\overline{\mathbb{D}}).$$

$\tilde{u}(w) = \tilde{f}(w)$ if $w \in \mathbb{T}$.

Define $u(z) := \tilde{u}(\hat{\varphi}(z))$.

$u \in Harm(\Omega)$ ($\tilde{u} = Re\,\tilde{F}$, $\tilde{F} \in A(\mathbb{D}) \Rightarrow \tilde{F} \circ \varphi \in A(\Omega) \Rightarrow$
$u = Re(\tilde{F} \circ \varphi) \in Harm(\Omega)$)

$u \in C(\overline{\Omega})$ (composition of continuous functions).

$z \in \partial\Omega \Rightarrow u(z) = \tilde{u}(\hat{\varphi}(z)) = \tilde{f}(\hat{\varphi}(z)) = f(z)$ – so it solves Dirichlet Problem!

$$\boxed{u(z) = \tilde{u}(\hat{\varphi}(z)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\hat{\varphi}(z)|^2}{|\hat{\varphi}(z)-e^{it}|^2}\, f(\hat{\varphi}^{-1}(e^{it}))\,dt}$$ ∎